VECTORS

BASIC RESULTS

2020 YIJC P1 Q1

Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{b} \neq \mathbf{0}$ and $3\mathbf{a} \times \mathbf{b} = 7\mathbf{b} \times \mathbf{c}$.

(i) Show that $3\mathbf{a} + 7\mathbf{c} = \lambda \mathbf{b}$, where λ is a constant. [2]

(ii) It is now given that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, that the modulus of $\mathbf{c}$ is $\frac{3}{7}$ and that the angle between $\mathbf{a}$ and $\mathbf{c}$ is $60°$.

Using a suitable scalar product, find exactly the two possible values of λ. [4]

Answer: $\lambda = \pm 3\sqrt{3}$

2019 EJC Promo Q4

Referred to the origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

(i) Given that non-zero numbers λ and μ are such that $\lambda \mathbf{a} + \mu \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $\lambda + \mu + 1 = 0$ with $\mu > 0$. Show that A, B, and C are collinear and find the ratio $CA : CB$ in terms of μ. [4]

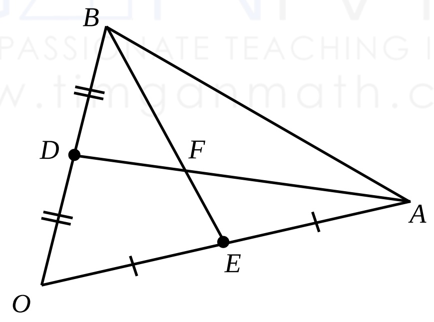
(ii) F is another point such that the line passing through A, B and C does not contain it. Find $\dfrac{\left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{AF} \times \overrightarrow{AC}\right|}$ in terms of μ. [2]

Answers: (i) $\mu : 1 + \mu$ (ii) $\dfrac{1+\mu}{\mu}$

2020 CJC P2 Q5 (a)

In the triangle shown below, one vertex is origin O, and the two other vertices are A and B where $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$.

A median of a triangle is a line segment joining a vertex $(O, A$ or $B)$ to the midpoint of the opposite side (e.g. AD is a median of triangle OAB from vertex A). It is given that F is the point of intersection between the medians of triangle OAB from vertices A and B.



(i) By finding $\overrightarrow{AD}$ and $\overrightarrow{BE}$, show that $\overrightarrow{OF}=\frac{1}{3}(\mathbf{a}+\mathbf{b})$. [3]

(ii) Prove that F also lies on OC, the median of triangle OAB from vertex O. [2]

Answer: (ii) Since $\overrightarrow{OF}=k\overrightarrow{OC}$ for some scalar k where $0<k<1$, point F lies on OC.

2021 TMJC P2 Q3

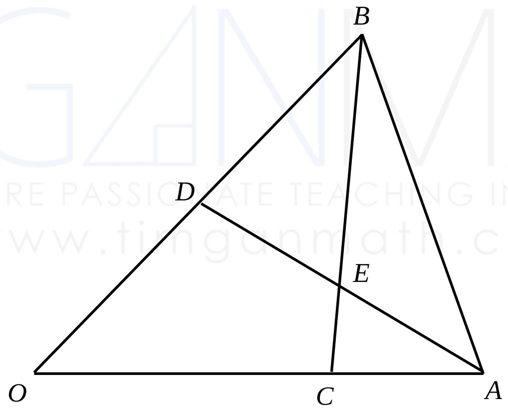


Figure 1

With reference to the origin O, the points A and B are such that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. Point D is the mid-point of OB and point C lies on OA such that $2OC=3CA$. The lines AD and BC intersect at point E (see Figure 1).

(i) Show that the vector equation of the line BC can be written as $\mathbf{r}=\frac{3}{5}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$, where λ is a real parameter. [1]

(ii) Find the position vector of E in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

(iii) Point F is such that $\overrightarrow{OF}$ is in the same direction as $\overrightarrow{AB}$. Given that the area of trapezium $OABF$ is $\frac{13}{16}|\mathbf{a}\times\mathbf{b}|$ units ^{2}, find the position vector of F in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

Answers: (ii) $\overrightarrow{OE}=\frac{3}{7}\mathbf{a}+\frac{2}{7}\mathbf{b}$ (iii) $\overrightarrow{OF}=\frac{5}{8}(\mathbf{b}-\mathbf{a})$

Prove that the medians of a triangle intersect one another at two-thirds the distance from the vertex to the mid-point of the opposite side.

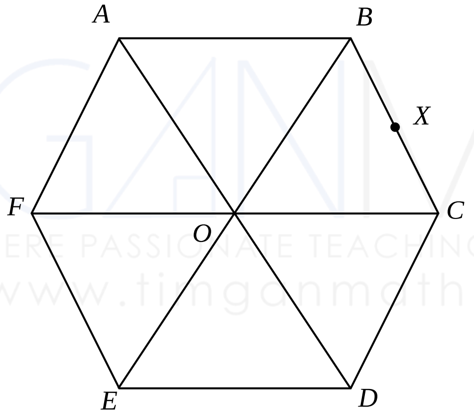
2019 TJC P1 Q9

(a) Vectors $\mathbf{u}$ and $\mathbf{v}$ are such that $\mathbf{u}\cdot\mathbf{v}=-1$ and $(\mathbf{u}\times\mathbf{v})+\mathbf{u}$ is perpendicular to $(\mathbf{u}\times\mathbf{v})+\mathbf{v}$.

Show that $|\mathbf{u}\times\mathbf{v}|=1$. [3]

Hence find the angle between $\mathbf{u}$ and $\mathbf{v}$. [3]

(b) The figure shows a regular hexagon $ABCDEF$ with O at the centre of the hexagon. X is the midpoint of BC.



Given that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$, find $\overrightarrow{OF}$ and $\overrightarrow{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

Line segments AC and FX intersect at the point Y. Determine the ratio $AY:YC$. [4]

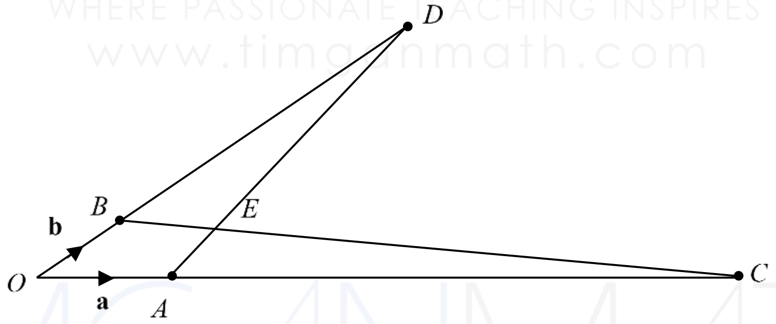
Answer: (a) $\frac{3}{4}\pi$ (b) $\overrightarrow{OF}=\mathbf{a}-\mathbf{b}$, $\overrightarrow{OX}=\mathbf{b}-\frac{1}{2}\mathbf{a}$, Ratio $AY:YC$ is $3:2$

2017 Specimen Paper P2 Q3

(a) The angle between the vectors $3\mathbf{i}-2\mathbf{j}$ and $6\mathbf{i}+d\mathbf{j}-\sqrt{7}\mathbf{k}$ is $\cos^{-1}\left(\frac{6}{13}\right)$.

Show that $2d^2-117d+333=0$. [3]

(b)



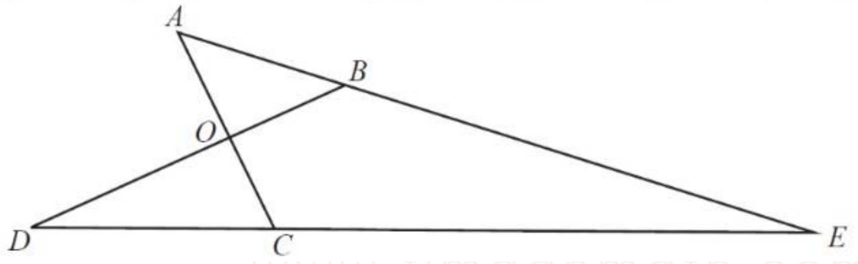
With reference to the origin O, the points A, B, C and D are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{AC}=5\mathbf{a}$ and $\overrightarrow{BD}=3\mathbf{b}$. The lines AD and BC cross at E (see diagram).

(i) Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [6]

(ii) The point F divides the line CD in the ratio $5:3$. Show that O, E and F are collinear, and find $OE:OF$. [4]

Answers: (b) (i) $\overrightarrow{OE}=\frac{18}{23}\mathbf{a}+\frac{20}{23}\mathbf{b}$ (ii) $\overrightarrow{OF}=\frac{9}{4}\mathbf{a}+\frac{5}{2}\mathbf{b}$; $8:23$

2017 TJC P2 Q4



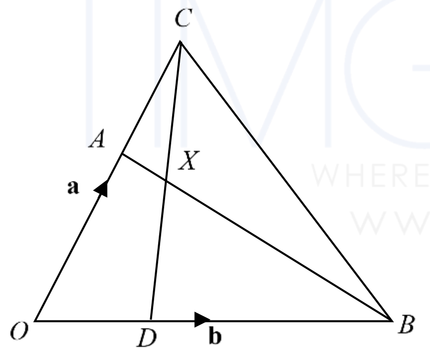
With reference to origin O, the points A, B, C and D are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=-\mathbf{a}$ and $\overrightarrow{OD}=-2b$. The lines AB and DC meet at E.

Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

Hence show that $\dfrac{BE}{AB}=3$. [1]

Answer: $\overrightarrow{OE}=4\mathbf{b}-3\mathbf{a}$

2017 AJC P1 Q9



With reference to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

C lies on OA produced with $OC:AC=3:1$ and D divides OB in a ratio of $2:3$.

X is the point of intersection of AB and CD.

(i) Show that $\overrightarrow{OX}=\dfrac{9}{11}\mathbf{a}+\dfrac{2}{11}\mathbf{b}$.

(ii) Hence, find the area of the triangle ADX, giving your answer in the form $k|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be determined.

Answer: (ii) $k=\dfrac{3}{55}$

2014 NYJC P1 Q5

Relative to the origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. The length of $\mathbf{a}$ is 2 units and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\pi}{6}$ radians. It is also given that the vector $\mathbf{a}$ and $\mathbf{b}-2\mathbf{a}$ are perpendicular. Find

(i) the exact length of projection of **a** onto **b**, [2]

(ii) the exact length of **b**. [3]

The point P lies on OB such that the ratio $OP:OB=3:4$. Find the exact area of triangle APB. [4]

Answers: (i) $\sqrt{3}$ (ii) $\dfrac{8}{\sqrt{3}}$ (iii) $\dfrac{1}{\sqrt{3}}$

2021 ACJC P2 Q2

Relative to the origin O, the points A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, such that $OACB$ is a quadrilateral. Let P, Q, R and S be the midpoints of the line segments OA, AC, CB and OB respectively.

(i) Show that $PQRS$ is a parallelogram. [2]

(ii) For any vectors $\mathbf{p}$ and $\mathbf{q}$, state the condition for $|\mathbf{p}+\mathbf{q}|=|\mathbf{p}|+|\mathbf{q}|$. [1]

(iii) Hence, by considering vector products, show that the area of $OACB$ is twice the area of $PQRS$. [3]

Answers: (ii) $|\mathbf{p}+\mathbf{q}|=|\mathbf{p}|+|\mathbf{q}|$ if and only if $\mathbf{p}$ and $\mathbf{q}$ are parallel and in the same direction.

(iii) $\dfrac{1}{2}|\mathbf{a}\times\mathbf{c}|+\dfrac{1}{2}|\mathbf{b}\times\mathbf{c}|=2$ (Area of $PQRS$)

Geometrical Interpretation

2015 JJC P1 Q7

The plane π has equation $13x-5y-7z=0$, and the line l_1 has equation $\frac{x-11}{3}=\frac{y-7}{5}=\frac{z-4}{2}$.

(i) Calculate $\begin{pmatrix}3\\5\\2\end{pmatrix}\bullet\begin{pmatrix}13\\-5\\-7\end{pmatrix}$ and state the geometrical relationship between the line l_1 and the plane π.

(ii) Given that P is the point $(11,7,4)$, find the acute angle between the line OP and the plane π.

(iii) Find the shortest distance from O to l_1.

(iv) The line l_2 has equation $\mathbf{r}=\lambda\mathbf{d},$ where $\mathbf{d}\neq\mathbf{0}$. Given that $\begin{pmatrix}3\\5\\2\end{pmatrix}\times\mathbf{d}=\mathbf{0},$ what can be deduced about the lines l_1 and l_2?

Answers: (ii) $22.1°$ (iii) $\sqrt{34}$

2020 CJC P2 Q5 (b)

With reference to the origin O, the points N, P and Q have position vectors $\mathbf{n}$, $\mathbf{p}$ and $\mathbf{q}$ respectively, and . $\prod$ .has an equation $\mathbf{r}\cdot\mathbf{n}=0$. It is known that $\mathbf{p}\cdot\mathbf{n}=\mathbf{q}\cdot\mathbf{n}\neq 0$.

(i) Show that $\overrightarrow{PQ}$ is perpendicular to $\mathbf{n}$. Hence describe the geometrical relationship between the line PQ and the plane $\prod$ [3]

(ii) Find a vector equation of the plane that contains points P and Q, and is perpendicular to plane $\prod$, leaving your answer in terms of $\mathbf{n}$, $\mathbf{p}$ and $\mathbf{q}$. [2]

Answer: (ii) $\mathbf{r}=\mathbf{p}+\lambda(\mathbf{q}-\mathbf{p})+\mu\mathbf{n},\ \lambda,\ \mu\in\mathbb{R}$

2019 TMJC P1 Q2

Referred to the origin O, A is a fixed point with position vector $\mathbf{a}$, and $\mathbf{d}$ is a non-zero vector. Given that a general point R has position vector $\mathbf{r}$ such that $\mathbf{r}\times\mathbf{d}=\mathbf{a}\times\mathbf{d}$, show that $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, where λ is a real constant. Hence give a geometrical interpretation of $\mathbf{r}$. [3]

Let $\mathbf{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{d}=\begin{pmatrix}2\\1\\5\end{pmatrix}$. By writing $\mathbf{r}$ as $\begin{pmatrix}x\\y\\z\end{pmatrix}$, use $\mathbf{r}\times\mathbf{d}=\mathbf{a}\times\mathbf{d}$ to form three equations which represent cartesian equations of three planes. State the relationship between these three planes. [3]

Answers: $\mathbf{r}$ is the position vector of a point on the line which passes through A and parallel to the vector $\mathbf{d}$.

Planes intersect at the line $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\5\end{pmatrix};x\in\mathbb{R}$

2019 VJC P1 Q5

Referred to the origin O, points P and Q have position vectors $3\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$ respectively. Point M is a point on QP extended such that $PM:QM$ is $2:3$.

(i) Find the position vector of point M in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) Find $\overrightarrow{PQ}\times\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

(iii) State the geometrical meaning of $\dfrac{\left|\overrightarrow{PQ}\times\overrightarrow{OM}\right|}{\left|\overrightarrow{PQ}\right|}$. [1]

2019 MI P1 Q8

(a) Referred to the origin O, the point Q has position vector $\mathbf{q}$ such that

$$\mathbf{q} = 2\mathbf{i} - \frac{3}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}.$$

(i) Find the acute angle between $\mathbf{q}$ and the y-axis. [2]

It is given that a vector $\mathbf{m}$ is perpendicular to the xy-plane and its magnitude is 1.

(ii) With reference to the xy-plane, explain the geometrical meaning of $|\mathbf{q}\cdot\mathbf{m}|$ and state its value. [2]

(b) Referred to the origin O, the point R has position vector $\mathbf{r}$ given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a positive constant and $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. It is known that $\mathbf{c}$ is a non-zero vector that is not parallel to $\mathbf{a}$ and $\mathbf{b}$. Given that $\mathbf{c}\times\mathbf{a} = \lambda\mathbf{b}\times\mathbf{c}$, show that $\mathbf{r}$ is parallel to $\mathbf{c}$. [2]

It is also given that $\mathbf{a}$ is a unit vector that is perpendicular to $\mathbf{b}$ and $|\mathbf{b}| = 2$.

By considering $\mathbf{r}\cdot\mathbf{r}$, show that $|\mathbf{c}| = k\sqrt{4\lambda^2+1}$, where k is a non-zero constant. [4]

Answers: (a)(i) 54.0° (or 0.942 rad) (ii) $\frac{1}{2}$ (b) $k = \sqrt{\frac{1}{\mu^2}}$ or $k = \frac{1}{|\mu|}$

2015 HCI P2 Q1

The position vectors of the fixed points A, B and C relative to the origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

(a) A variable point R has position vector $\mathbf{r}$. Describe the locus of R if $\mathbf{r}\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$. [2]

(b) Given that $\mathbf{a} = (2\mathbf{a}\cdot\mathbf{b})\mathbf{b}$, state the geometrical relation between $\mathbf{a}$ and $\mathbf{b}$, and find $|\mathbf{b}|$. [3]

(c) A point N divides AB in the ratio $2:1$ and M is the mid-point of BC. Given that O is the mid-point of MN, show that $2\mathbf{a} + 7\mathbf{b} + 3\mathbf{c} = \mathbf{0}$. [3]

Answer: (ii) $|\mathbf{b}| = \frac{1}{\sqrt{2}}$

2015 DHS P1 Q7

Relative to the origin O, two points A and B have position vectors **a** and **b** respectively.

It is given that $|\mathbf{a}|=2$, $|\mathbf{b}|=1$ and $|3\mathbf{a}-2\mathbf{b}|=\sqrt{(37)}$.

(i) By considering the scalar product $(3\mathbf{a}-2\mathbf{b})\cdot(3\mathbf{a}-2\mathbf{b})$, show that $\mathbf{a}\cdot\mathbf{b}=\frac{1}{4}$ and give the geometrical meaning of $|\mathbf{a}\cdot\mathbf{b}|$. [4]

(ii) Give the geometrical meaning of $|(\mathbf{a}-\mathbf{b})\times\mathbf{b}|$ and find its exact value. [3]

(iii) Write down, in terms of **a** and **b**, a vector equation of the line that passes through O and bisects the angle AOB. [1]

Answers: (ii) $\frac{3\sqrt{7}}{4}$ (iii) $\mathbf{r}=\lambda(\mathbf{a}+2\mathbf{b}),\lambda\in\mathbb{R}$

2019 SAJC Promo Q4

Relative to the origin O, the points A and B have position vectors **a** and **b** respectively. It is given that the magnitude of **a** is 4 and **b** is a unit vector perpendicular to **a**.

(i) Find the value of $(2\mathbf{a}+\mathbf{b})\cdot(3\mathbf{a}-5\mathbf{b})$. [1]

(ii) The point C is on AB such that $AC:CB=3:1$. Write down the position vectors of C, **c**, in terms of **a** and **b**. [1]

(iii) State the geometrical meaning of $|\mathbf{b}\times\mathbf{c}|$ and find its exact value. [5]

(i) 91 (ii) $\frac{\mathbf{a}+3\mathbf{b}}{4}$ (iii) 1

2016 RI Promo Q2

The variable vector **v** satisfies the equations $\mathbf{v}\cdot(2\mathbf{j}+\mathbf{k})=\mathbf{0}$ and $\mathbf{v}\cdot(3\mathbf{i}+7\mathbf{k})=\mathbf{0}$.

(i) Find the set of vectors **v**. [2]

(ii) Describe geometrically the set of points with position vectors found in part **(i)**. [1]

Answers: (i) $\mathbf{v}\in\left\{\mu\begin{pmatrix}14\\3\\-6\end{pmatrix}\middle|\mu\in\mathbb{R}\right\}$

2019 NJC Promo Q9

Non-zero and non-parallel vectors **a**, **b** and **c** are such that $\mathbf{b}\times3\mathbf{c}=\mathbf{c}\times\mathbf{a}$.

(i) Determine the relationship between **c** and $\mathbf{a}+3\mathbf{b}$, justifying your answer. [2]

It is given that **a** and $3\mathbf{b}$ are unit vectors and that the angle between **a** and **b** is $60°$.

(ii) Evaluate $|\mathbf{a}+3\mathbf{b}|$. [3]

(iii) Given further that $\mathbf{a}+3\mathbf{b}$ makes an angle of $60°$, $120°$ and $135°$ with the positive x-, y- and z-axes respectively, show that **c** is parallel to $\mathbf{i}-\mathbf{j}-\sqrt{2}\mathbf{k}$. [3]

Answer: (ii) $\sqrt{3}$

2020 ACJC P2 Q1

(a) If $|\mathbf{a}|=2$, $|\mathbf{a}-\mathbf{b}|=\sqrt{3}$ and $\mathbf{a}\cdot\mathbf{b}=\frac{5}{2}$, use the cosine rule for triangles to find $|\mathbf{b}|$. [3]

(b) It is given that $\mathbf{r}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ where a,b and c are constants.

Give the geometrical interpretations of

(i) $|\mathbf{r}\cdot\mathbf{k}|$, [1]

(ii) $|\mathbf{r}\times\mathbf{k}|$, [1]

(iii) Given that $\mathbf{k}\times\mathbf{r}=\mathbf{p}$ and $\mathbf{r}\times\mathbf{p}=\mathbf{k}$, in either order, show that $a^2+b^2=1$ and find the value of c. [4]

Answers: (a) 2 (b)(i) Length of projection of **r** onto the z-axis or shortest distance of point $\begin{pmatrix}a\\b\\c\end{pmatrix}$ to the xy plane. (ii) Area of parallelogram with adjacent sides given by vectors **r** and **k** (iii) $c=0$

Planes and Lines

2015 IJC P1 Q11

The plane p_1 has equation $x-2y+3z=5$. The plane p_2 contains the points A and B with coordinates $(1,-3,1)$ and $(4,3,-2)$ respectively and is perpendicular to p_1.

(i) Find the cartesian equation of p_2. [3]

(ii) Find the exact perpendicular distance from A to p_1. [2]

(iii) Find the vector equation of the line l where p_1 and p_2 meet. [2]

(iv) Given that C is a general point on l find an expression for the square of the distance BC. Hence, or otherwise, find the position vector of the point on l which is nearest to B. [5]

Answers: (i) $x-y-z=3$ (ii) $\frac{5}{\sqrt{14}}$ (iii) $\mathbf{r}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}5\\4\\1\end{pmatrix}, \lambda\in\mathbb{R}$ (iv) $42\lambda^2-66\lambda+38$; $\frac{1}{14}\begin{pmatrix}69\\16\\11\end{pmatrix}$

2021 SAJC P2 Q3

The line L has equation $\mathbf{r}=-\mathbf{i}+2\mathbf{j}-4\mathbf{k}+\lambda(6\mathbf{i}+3\mathbf{j}-2\mathbf{k})$, where $\lambda\in\mathbb{R}$.

(i) Find the acute angle between L and the x-axis. [2]

The point P has position vector $6\mathbf{i}+3\mathbf{j}-5\mathbf{k}$.

(ii) Find the points on L which are at a distance of $\sqrt{59}$ units from P.
Hence or otherwise find the point on L which is closest to P. [5]

(iii) Find a cartesian equation of the plane that includes the line L and the point P. [3]

Answers: (i) 31.0° (ii) The points are $(11, 8, -8)$ or $\left(-\frac{73}{49},\frac{86}{49},-\frac{188}{49}\right)$. Hence, the point closest to P is $\left(\frac{233}{49},\frac{239}{49},-\frac{290}{49}\right)$

(iii) $x+8y+15z=-45$

2021 NYJC P2 Q1

The plane $\prod_1$ passes through $(3,-1,2)$ and is perpendicular to the line $\mathbf{r} = 2\mathbf{i}+4\mathbf{j}-3\mathbf{k}+\lambda(5\mathbf{i}+2\mathbf{j}-\mathbf{k})$. The plane $\prod_2$ contains the points $(2,3,2)$, $(4,1,-1)$ and $(0,-1,2)$.

(i) Show that the acute angle, θ, between the planes $\prod_1$ and $\prod_2$ is such that $\cos\theta = \frac{\sqrt{30}}{15}$. [3]

(ii) Show that the line of intersection, L, of the planes $\prod_1$ and $\prod_2$ has vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, $\mu \in \mathbb{R}$. [3]

The plane $\prod_3$ has the equation $4(k-2)x+(k+1)y-4k^2z=8$, where k is a constant.

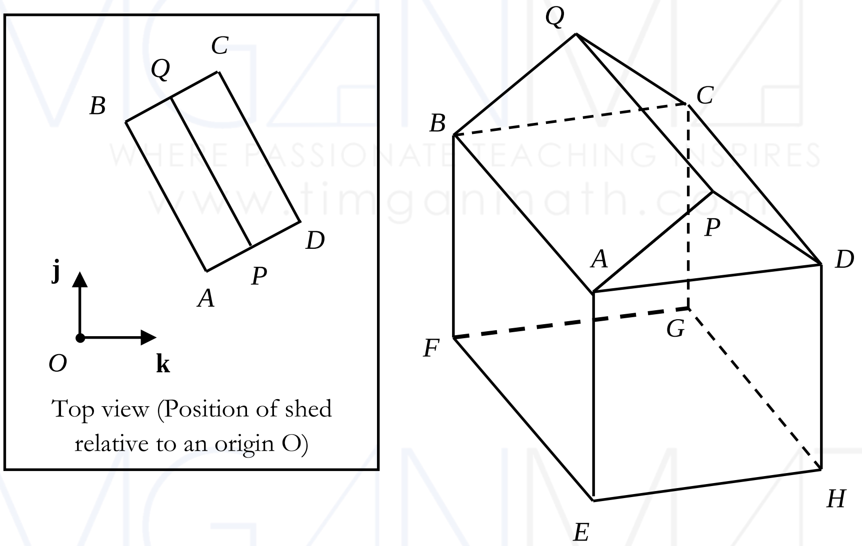
(iii) The three planes $\prod_1$, $\prod_2$ and $\prod_3$ have no points in common. By considering the relationship between the line L and the plane $\prod_3$, find the possible values of k. [2]

(iv) For the positive value of k found in (iii), find the distance between L and $\prod_3$. [2]

Answers: (iii) $k=\pm 1$ (iv) 3

2021 ACJC P1 Q11

Mr Neo wants to build a shed in his future garden as shown in the diagrams below.



The planes $ABCD$ and $EFGH$ are parallel to the horizontal plane, represented by the xy -plane. The triangles APD and BQC are congruent isosceles triangles and the lengths of the pillars AE, BF, CG and DH are of the same height.

The equation of the plane $ABQP$ is given by $-2x-y+3z=2$ and the point P has position vector $4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$.

(i) Explain why plane $CDPQ$ is perpendicular to $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$. Hence show that the equation of the plane $CDPQ$ is $2x+y+3z=22$. [2]

(ii) Find the angle the roof $ABQP$ makes with the horizontal plane. [2]

(iii) Find the vector equation of the line PQ and hence find the coordinates of the point Q given that PQ has length $3\sqrt{5}$ units. [5]

(iv) When the shed was completed, Mr Neo discovered a hole in the roof $ABQP$. When light shines perpendicularly onto the plane $ABQP$, the light passes through the hole and hits the ground at the point with coordinates $(3,6,0)$.
Find the coordinates of the hole in the roof $ABQP$. [3]

Answers: (i) $ABQP$ and $CDPQ$ are symmetric about the vertical plane passing through PQ. Hence direction of the normal is $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$. (ii) 36.7° (iii) $\begin{pmatrix}1\\8\\4\end{pmatrix}$ or $\begin{pmatrix}7\\-4\\4\end{pmatrix}$ (reject, from top view y -coordinate >2) (iv) $(1,5,3)$

2021 TJC P1 Q11

Coordinates axes $Oxyz$ are set up with the origin O at the base of an airport control tower. The x-axis is due East, the y-axis due North and the z-axis vertical. The units of distances are kilometres. An airplane A takes off from the point X. For the first 4 minutes, the position vector of A at time t minutes after take-off, is given by

$$\mathbf{r}=(2+t)\mathbf{i}+(1+2t)\mathbf{j}+3t\mathbf{k}, \qquad 0\le t\le 4.$$

(i) State the coordinates of X. [1]

(ii) Find the acute angle the flight path makes with the horizontal. [2]

(iii) The airplane enters a cloud at a height of 5 km. Find the coordinates of the point where it enters the cloud. [2]

A second airplane B takes off from the point $(-2,-1,0)$ at the same time as the first airplane A and is traveling at a constant speed in a straight line for the first 4 minutes. Two minutes after take-off, B is at the point $(1,5,\alpha)$.

(iv) Find in terms of α, the position vector of B after t minutes where $0\le t\le 4$. Explain if it is possible for the two airplanes to collide in the first 4 minutes. [4]

(v) At $t=4$, a third airplane C was spotted to be equidistant from the first two airplanes. At the same instant, two buildings on the ground D and E are such that A and B are equidistant from both D and E, i.e. $AD=BD$ and $AE=BE$. Find the Cartesian equation of the plane in terms of α, in which C, D and E lie. [4]

Answers: (i) $(2,1,0)$ (ii) 53.3° (iii) $\left(\frac{11}{3},\frac{13}{3},5\right)$ (iv) $t\begin{pmatrix}\frac{1}{2}\\1\\\frac{a}{2}-3\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$ There is no solution to the above equation. Hence the two airplanes will not collide. (v) $-x+y+(\alpha-6)z=a^2-31$

N2003 P1 Q5

Referred to the origin O, the position vectors of points A and B are $4\mathbf{i}-11\mathbf{j}+4\mathbf{k}$ and $7\mathbf{i}+\mathbf{j}+7\mathbf{k}$ respectively.

(i) Find a vector equation for the line l passing through A and B. [2]

(ii) Find the position vector of the point P on l such that OP is perpendicular to l. [4]

Answers: (i) $(4\mathbf{i}-11\mathbf{j}+4\mathbf{k})+\lambda(\mathbf{i}+4\mathbf{j}+\mathbf{k}), \lambda\in\mathbb{R}$, (ii) $(6,-3,6)$

PJC 2015 P1 Q12

Do not use a graphing calculator in answering this question.

The planes p_1 and p_2 have equations $2x-z=3$ and $x-3y=-3$ respectively. The point A with position vector $\lambda\mathbf{i}+\mathbf{j}+\mu\mathbf{k}$, where λ and μ are constants, is in both p_1 and p_2.

(i) Find the values of λ and μ. [2]

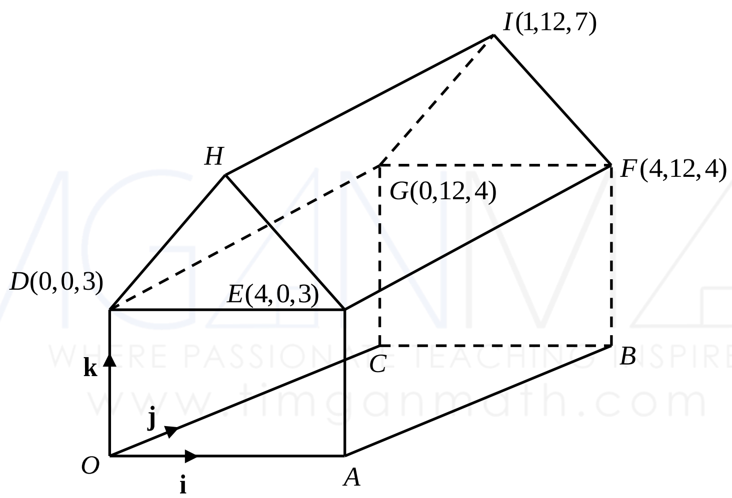
(ii) The planes p_1 and p_2 intersect in a line l. Find a vector equation of l. [2]

(iv) The point B has position vector $\mathbf{i}+2\mathbf{j}+4\mathbf{k}$. Find the position vector of N, the foot of the perpendicular from B to p_1. Hence find the vector equation of the line of reflection of AB in p_1. [5]

Answers: (i) $\lambda=0, \mu=-3$ (ii) $\mathbf{r}=\begin{pmatrix}0\\1\\-3\end{pmatrix}+t\begin{pmatrix}3\\1\\6\end{pmatrix}, t\in\mathbb{R}$ (iv) $\overrightarrow{ON}=\begin{pmatrix}3\\2\\3\end{pmatrix}$; $\mathbf{r}=\begin{pmatrix}0\\1\\-3\end{pmatrix}+\phi\begin{pmatrix}5\\1\\5\end{pmatrix}, \phi\in\mathbb{R}$

2020 CJC P1 Q11

A temporary isolation centre is built to manage the increasing number of COVID-19 cases. The roof takes the shape of a triangular prism. Points (x, y, z) are defined relative to an origin, O, with unit vectors $\mathbf{i}$ along $\overrightarrow{OA}$, $\mathbf{j}$ along $\overrightarrow{OC}$, and $\mathbf{k}$ along $\overrightarrow{OD}$ (see diagram). The coordinates of D, E, F, G and I are $(0,0,3)$, $(4,0,3)$, $(4,12,4)$, $(0,12,4)$ and $(1,12,7)$ respectively. The units are measured in metres.



(i) Find a cartesian equation of the plane that contains the roof section $EFIH$. [3]

It is given that the roof section $DGIH$ is part of the plane with equation $36x+y-12z=-36$.

(ii) Find a cartesian equation of the line that contains the roof ridge HI. [3]

Steel cables are used to hold the isolation centre in place. Cables are laid in straight lines and the widths of cables can be neglected. It is given that cable 1 passes through F and D and cable 2 passes through G and M, where M is the mid point of EF.

A builder needs to locate the point J where both cables meet.

(iii) Find the coordinates of J. [4]

To strengthen the structure, it is recommended that another steel cable should be extended from J to the closest point on the roof ridge HI. The builder is left with 3.2 metres of steel cable.

(iv) Determine whether the remaining steel cable is long enough to connect J to the roof ridge HI. [4]

Answers: (i) $12x-y+12z=84$ (ii) Cartesian equation for the line $l: x-1, \frac{y+72}{12}=z$ (iii) $\left(\frac{8}{3}, 8, \frac{11}{3}\right)$

2016 PJC J1 CT Q10

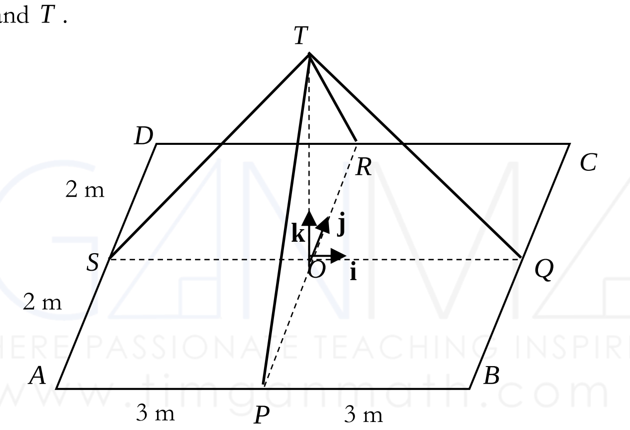
During a camp, a group of scouts builds a structure using wooden poles as shown in the diagram. The structure has a rectangular base $ABCD$ on the horizontal ground, with $AB = 6$ metres and $AD = 4$ metres. The top of the structure, T, is 2 metres vertically above O, the centre of $ABCD$. T is connected to P, Q, R and S, the mid-points of AB, BC, CD and DA respectively.

(i) If $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of OQ, OR and OT respectively, write down the position vectors of Q, R and T. Hence find angle QTR to the nearest degree.

(ii) The scouts tie a rope from P to a point E on ST such that $4\overrightarrow{SE} = \overrightarrow{ST}$. Find the length of the rope between E and P, assuming that the rope is taut.

(iii) Another rope is tied from P to a point F on TQ. Find the position vector of F such that the length of the rope between F and P is minimised.

(iv) A flag pole of 5 metres is erected vertically at the point $G(\alpha, \beta, 0)$. Find the values of α and β if the top of the flag pole is collinear with A and T.



Answers: (i) $66.9°$ (ii) $\dfrac{\sqrt{149}}{4}$ units (iii) $\dfrac{1}{13}\begin{pmatrix}12\\0\\18\end{pmatrix}$ (iv) $\alpha = \dfrac{9}{2}$, $\beta = 3$

N2020 P1 Q5

(a) Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$, find the relationship between $\mathbf{a}$ and $\mathbf{b}$. [2]

(b) The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively. The points P and Q are fixed and R varies. [3]

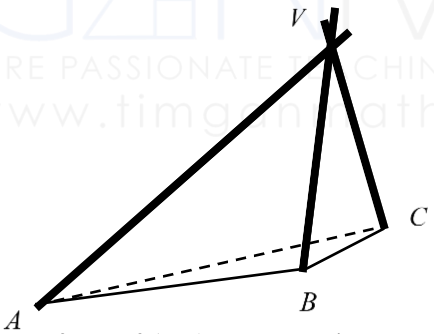
(i) Given that $\mathbf{q}$ is non-zero and $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$, describe geometrically the set of all possible positions of the point R. [3]

(ii) Given instead that $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{p} = \begin{pmatrix}-1\\2\\4\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}3\\-5\\2\end{pmatrix}$ and that $(\mathbf{r} - \mathbf{p}) \cdot \mathbf{q} = 0$, find the relationship between x, y and z. Describe the set of all possible positions of the point R in this case. [4]

Answers: (a) $\mathbf{a}$ is parallel to $\mathbf{b}$. (b)(i) H is a line which passes thought point P and parallel to the vector $\mathbf{q}$. (ii) Point R is on the plane which has a normal vector $\begin{pmatrix}3\\-5\\2\end{pmatrix}$ and contains point P.

2020 TJC P1 Q9

A camper built a tripod tent using 3 poles and 2 canvases as shown in the diagram below. The 3 poles rest on **sloping ground** at the points A, B, and C, and are fastened together at the point V. With respect to the origin (not shown in the diagram), A, B and C have position vectors and $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$, $7\mathbf{i}+4\mathbf{j}$ and $8\mathbf{i}+12\mathbf{j}+2\mathbf{k}$ respectively, where $\mathbf{k}$ is perpendicular to the horizontal. You may assume that the sloping ground is a plane.



(i) Find a vector equation, in parametric form, of the sloping ground. [2]

(ii) The vectors $\overrightarrow{AV}$ and $\overrightarrow{CV}$ are parallel to $4\mathbf{i}+\alpha\mathbf{j}+10\mathbf{k}$ and $-2\mathbf{i}-2\mathbf{j}+10\mathbf{k}$ respectively. Find, in either order, the value of α and the position vector of V. [4]

(iii) Find the acute angle between the pole CV and the sloping ground. [3]

(iv) The camper attaches a small torchlight to the top of the tent at V. The torchlight was not fastened properly and dropped vertically down to the ground. Assuming the lowest end of the torch is at V, find the distance travelled by the torchlight from V to the ground. [3]

Answers: (i) $\mathbf{r}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+s\begin{pmatrix}5\\1\\1\end{pmatrix}+t\begin{pmatrix}2\\3\\1\end{pmatrix}$, $s,t\in\mathbb{R}$ (ii) $\alpha=\frac{74}{11}$; Coordinates of V are $\left(\frac{32}{5},\frac{52}{5},10\right)$ (iii) $86.9°$ (iv) 8.62 (3 s.f.)

2018 ACJC P1 Q11

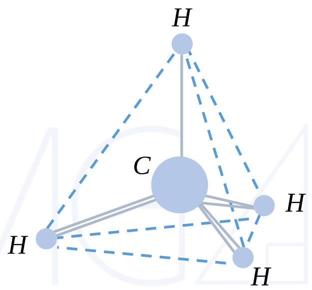


Figure 1

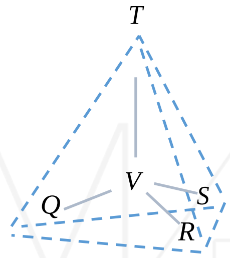


Figure 2

Figure 1 shows a methane molecule consisting of a carbon atom with four hydrogen atoms symmetrically placed around it. Figure 2 shows the tetrahedron structure of the methane molecule with the centre of the hydrogen atoms represented by points Q, R, S and T and the centre of the carbon atom represented by point V.

The points Q, R and S has coordinates $(8,1,8)$, $(8,7,2)$ and $(2,1,2)$ respectively and form an equilateral triangle.

(i) Find a cartesian equation of the plane p which passes through the point Q, R and S. [4]

(ii) Find a cartesian equation of the plane p_1 which passes through the midpoint of QR and is perpendicular to QR. [2]

Plane p_2 which passes through the midpoint of RS and is perpendicular to RS has equation $x+y=9$.

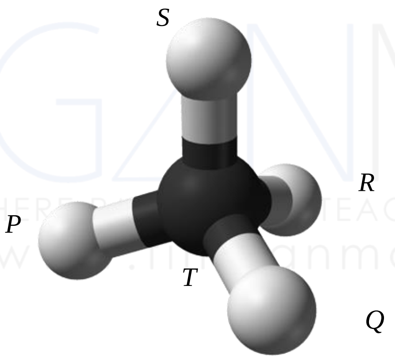
(iii) Find a vector equation of the line l where p_1 and p_2 meet. [1]

(iv) The point T is on the line l such that $QRST$ is a regular tetrahedron with $QR=QT$.
Show that the possible coordinates for the point T is $(2,7,8)$. Hence, or otherwise, find the coordinates of a point on plane p that is the closest to point T. [5]

(v) Given that TV is $\frac{3}{4}$ of the vertical height of tetrahedron $QRST$. Find the coordinates of point V and hence show that the bonding angle TVQ of the methane molecule is $109.5°$ (correct to 1 decimal place). [4]

Answers: (i) $-x+y+z=1$ (ii) $y-z=-1$ (iii) $l:\mathbf{r}=\begin{pmatrix}10\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix};\lambda\in\mathbb{R}$ (iv) $(6,3,4)$ (v) $(5,4,5)$

2019 EJC P1 Q11



Methane $(\mathrm{CH_4})$ is an example of a chemical compound with a tetrahedral structure. The 4 hydrogen (H) atoms form a regular tetrahedron, and the carbon (C) atom is in the centre.

Let the 4 H - atoms be at points P, Q, R, and S with coordinates $(9,2,9)$, $(9,8,3)$, $(3,2,3)$, and $(3,8,9)$ respectively.

(i) Find a Cartesian equation of the plane $\prod_1$ which contains the points P, Q and R. [4]

(ii) Find a Cartesian equation of the plane $\prod_2$ which passes through the midpoint of PQ and is perpendicular to $\overrightarrow{PQ}$. [2]

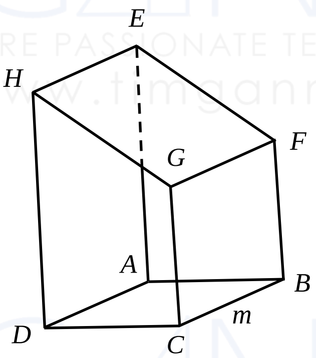
(iii) Find the coordinates of point F, the foot of the perpendicular from S to $\prod_1$. [4]

(iv) Let T be the point representing the carbon (C) atom. Given that point T is perpendicular from the points P, Q, R, and S, find the coordinates of T. [3]

Answers: (i) $-x+y+z=z$ (ii) -1 (iii) $\begin{pmatrix}7\\4\\5\end{pmatrix}$ (iv) $\overrightarrow{OT}=\begin{pmatrix}6\\5\\6\end{pmatrix}$

2018 ACJC MYE Q7

The diagram shows the structure of a building with a horizontal rectangular base $ABCD$, whereby BC is m units. The structure consists of four vertical columns, AE, DH, CG, and BF which are parallel to the z - axis.
The roof $EFGH$ has equation $x-y+\sqrt{2}z=2+4\sqrt{2}$ and the plane $ABFE$ has equation $x+y=2$. The origin $(0,0,0)$ is a point within the rectangular base $ABCD$.



(i) Find the cartesian equation of the plane $DCGH$ in terms of m. [2]

(ii) Find the acute angle between the roof $EFGH$ and the base $ABCD$. [2]

To ensure that the roof is sturdier, a beam needs to be added to connect point $W(0,2,4)$, which lies on column AE, to the nearest point on the roof.

(iii) Show that this nearest point of the roof is $\left(1,1,4+\sqrt{2}\right)$. [3]

(iv) Hence find the exact cartesian equation of the beam EF. [3]

Ans: (i) $x+y=2-m\sqrt{2}$, where $m>\sqrt{2}$ (ii) 45 (iii) $N\left(1,1,4+\sqrt{2}\right)$ (iv) $x-1=1-y=\dfrac{4+\sqrt{2}-z}{\sqrt{2}}$